

Self-Intersections of a Closed Double Fermat Spiral

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Abstract

We study a closed two-arc curve constructed from the Fermat spiral and count its interior geometric self-intersections. The proposed main theorem is

$$N(T) = [5 / (12\pi^2)] \times T^2 + O(T^{7/4}).$$

Consequently, $N(T) / T^2$ tends to $5 / (12\pi^2) \approx 0.04221715985$. The leading constant decomposes as

$$5 / (12\pi^2) = 2 \times (5/6) \times [1 / (4\pi^2)],$$

corresponding to two triangle orientations, the normalized admissible area, and the density of the phase lattice. This manuscript is presented for independent verification; bibliographic priority is not claimed.

1. Definition of the curve

Identify the Euclidean plane with the complex plane. Define the Fermat spiral by

$$\gamma(t) = \sqrt{t} \times e^{it}, \quad t \geq 0.$$

For $T > 0$ define the second arc by

$$\eta_T(t) = \gamma(T) - \gamma(t), \quad 0 \leq t \leq T.$$

The closed double Fermat spiral is

$$C_T = \gamma([0, T]) \cup \eta_T([0, T]).$$

Each individual arc is injective because $|\gamma(t)| = \sqrt{t}$ determines t uniquely. Thus every nontrivial self-intersection is a crossing between the two arcs. $N(T)$ denotes the number of interior geometric crossing points, excluding the two prescribed joining points.

2. Exact intersection equation

An intersection occurs exactly when

$$\gamma(t) = \eta_T(s),$$

equivalently

$$\gamma(t) + \gamma(s) = \gamma(T). \quad (1)$$

Taking squared absolute values yields the exact relation

$$s = T + t - 2\sqrt{Tt} \cos(T - t). \quad (2)$$

Equation (1), together with its phase condition, is the fundamental equation of the problem.

3. Normalized geometric domain

Set $x = t/T$ and $y = s/T$. Equation (1) says that vectors of lengths $\sqrt{x}$, $\sqrt{y}$ and 1 must close to form a triangle. Hence

$$\sqrt{x} + \sqrt{y} \geq 1. \quad (3)$$

The admissible normalized domain is $D = \{(x,y) \text{ in } [0,1]^2 : \sqrt{x} + \sqrt{y} \geq 1\}$. Its complement has area $1/6$, so

$$\text{Area}(D) = 5/6. \quad (4)$$

4. Phase lattice

For an interior nondegenerate triangle, let $\alpha(x,y)$ and $\beta(x,y)$ be the relevant triangle angles. They satisfy

$$\cos \alpha = (1 + x - y) / (2\sqrt{x}),$$

$$\cos \beta = (1 + y - x) / (2\sqrt{y}).$$

There are two mirror orientations, $\sigma = +1$ and $\sigma = -1$. The phase equations take the form

$$T(x - 1) = 2\pi m + \sigma\alpha, \quad (5)$$

$$T(y - 1) = 2\pi n - \sigma\beta. \quad (6)$$

Thus the solutions are controlled asymptotically by a square lattice of spacing $2\pi/T$ and density

$$T^2 / (4\pi^2). \quad (7)$$

5. Regular core

Choose $\varepsilon = T^{-1/4}$ and remove a thin neighborhood of the degenerate boundary. On the remaining core, $x \geq \varepsilon$, $y \geq \varepsilon$ and $\sqrt{x} + \sqrt{y} \geq 1 + \varepsilon$.

Differentiating the angle formulas gives a uniform estimate

$$|\nabla\alpha| + |\nabla\beta| = O(\varepsilon^{-5/2}).$$

Define the phase map

$$\Phi_{-}(\sigma, T)(x, y) = (x - \sigma\alpha/T, y + \sigma\beta/T).$$

Then

$$D\Phi_{-}(\sigma, T) = I + O(T^{-3/8}). \quad (8)$$

Hence, for sufficiently large T , the phase map is injective and bi-Lipschitz on the regular core.

6. Counting the regular intersections

The phase lattice has cell area $4\pi^2/T^2$. The normalized admissible area tends to $5/6$. For one orientation the leading contribution is

$$[5 / (24\pi^2)] \times T^2.$$

There are two orientations, so the regular region contributes

$$N_{\text{regular}}(T) = [5 / (12\pi^2)] \times T^2 + O(T^{7/4}). \quad (9)$$

The leading coefficient is therefore

$$2 \times (5/6) \times [1 / (4\pi^2)] = 5 / (12\pi^2).$$

7. No-arc lemma

To control exceptional cells, we need to exclude a continuous overlap of the two arcs. The curvature of the Fermat spiral is

$$\kappa(t) = 2\sqrt{t(4t^2 + 3)} / (4t^2 + 1)^{3/2}.$$

Its derivative is

$$\kappa'(t) = -(16t^4 + 40t^2 - 3) / [\sqrt{t(4t^2 + 1)}^{5/2}].$$

The numerator has exactly one positive zero. Thus κ increases to a unique maximum and then decreases.

If $\gamma(t) + \gamma(s(t))$ were constant on a nontrivial interval, differentiation would impose compatible tangent directions and

$$\kappa(t) = \kappa(s(t)),$$

followed by

$$\kappa'(t) = s'(t) \kappa'(s(t)).$$

On the same monotonic side of the unique maximum, equality of curvature forces $t = s(t)$, contradicting the required tangent relation. On opposite sides, $\kappa'(t)$ and $\kappa'(s(t))$ have opposite signs while the tangent relation requires $s'(t) > 0$. Therefore no nontrivial common arc exists.

8. Uniform multiplicity in a phase cell

Write $t = 2\pi m + \xi$ and $s = 2\pi n + \eta$, with $0 \leq \xi, \eta < 2\pi$. Introduce

$$A = \sqrt{(t/T)}, \quad B = \sqrt{(s/T)}.$$

The intersection equations become

$$A^2 = 2\pi m/T + \xi/T,$$

$$B^2 = 2\pi n/T + \eta/T,$$

$$A \cos \xi + B \cos \eta = \cos T,$$

$$A \sin \xi + B \sin \eta = \sin T. \quad (10)$$

All variables lie in compact ranges: $0 \leq A, B \leq 1$ and $0 \leq \xi, \eta \leq 2\pi$. After a fixed finite subdivision of the angular domain, these equations form semi-Pfaffian sets of uniformly bounded format.

Bounds for Betti numbers of semi- and sub-Pfaffian sets give a uniform bound on the number of connected components. By the no-arc lemma, the relevant components are zero-dimensional. Therefore there exists an absolute constant C , independent of T , m and n , such that

$$\text{number of intersections in any phase cell} \leq C. \quad (11)$$

9. Exceptional region

The removed normalized region has area $O(T^{-1/4})$. Since the phase-lattice density is proportional to T^2 , it meets $O(T^{7/4})$ phase cells, up to lower-order boundary terms. By (11),

$$N_{\text{exceptional}}(T) = O(T^{7/4}). \quad (12)$$

10. Main theorem

Combining the regular and exceptional contributions gives the proposed theorem.

$$N(T) = [5 / (12\pi^2)] \times T^2 + O(T^{7/4}).$$

In particular,

$$\lim_{T \rightarrow \infty} N(T)/T^2 = 5 / (12\pi^2).$$

Numerically,

$$5 / (12\pi^2) \approx 0.04221715985.$$

Thus, to leading order,

$$N(T) \approx 0.04221716 \times T^2.$$

11. Additional critical parameters

The geometry also contains distinguished bifurcation values. Symmetric configurations lead to

$$T_k = 8\pi k / 3.$$

Boundary tangencies satisfy

$$\sin T + 2T \cos T = 0,$$

equivalently, when $\cos T \neq 0$,

$$\tan T = -2T.$$

A full classification of these critical parameters is a natural continuation of the present work.

12. Novelty and status

To the author's knowledge, the specific closed double Fermat spiral studied here and the stated asymptotic self-intersection law were not found in the literature examined so far. Related literature studies Fermat spirals in continuous-curvature paths, connected toolpaths, mirrored spiral segments and other geometric settings.

No claim of absolute bibliographic priority is made. This preprint is intended first for independent mathematical review. The proof has been internally audited, but independent verification and peer review remain distinct steps.

References

A. Gabriellov, N. Vorobjov and T. Zell, "Betti Numbers of Semialgebraic and Sub-Pfaffian Sets," Journal of the London Mathematical Society 69 (2004), 27–43. DOI: 10.1112/S0024610703004939.

T. Zell, Quantitative Study of Semi-Pfaffian Sets, 2004.

Further references on Fermat-spiral toolpaths, continuous-curvature paths and mirrored Fermat-spiral constructions should be added after the final bibliographic review.